Computing E-polynomials

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E-polynomials

The *Hodge polynomial* of a smooth projective variety X over $\mathbb{C}$ is

$$P(X) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X)\, u^p v^q$$

with $h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$ the *Hodge numbers* of X. It satisfies

- *scissor relation*: $P(X) = P(Z) + P(X \backslash Z)$ for $Z \subset X$ a closed subvariety
- *multiplicativity*: $P(X \times Y) = P(X) \cdot P(Y)$

This polynomial extends uniquely to any variety over $\mathbb{C}$ via the Grothendieck ring

$$e : K(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}[u, v]$$

where $e(X)$ is called the *E-polynomial* of X. Its coefficients are given by the *mixed Hodge numbers*

$$h^{p,q;k}_{\text{mixed}}(X) = \dim \mathrm{Gr}^p_F \mathrm{Gr}^W_{p+q} H^k_c(X, \mathbb{C})$$

of the mixed Hodge structure on the compactly supported cohomology of X [1].

Examples

- $e(\mathbb{A}^1) = e(\mathbb{P}^1) - e(\mathrm{pt}) = uv =: q$, the *Lefschetz motive*
- $e(\mathbb{P}^n) = e(\mathbb{A}^n) + e(\mathbb{A}^{n-1}) + \cdots + e(\mathbb{A}^1) + e(\mathbb{A}^0)$
  $= q^n + q^{n-1} + \cdots + q + 1$
- To compute $e(\mathrm{SL}(2, \mathbb{C})) = e(\{ad - bc = 1\})$, decompose

$$\mathrm{SL}(2, \mathbb{C}) = \left\{ a = 0, b \neq 0, c = \frac{-1}{b} \right\} \sqcup \left\{ a \neq 0, d = \frac{bc + 1}{a} \right\}$$

to find $e(\mathrm{SL}(2, \mathbb{C})) = \underset{(d)}{q}\, \underset{(b)}{(q-1)} + \underset{(b,c)}{q^2}\, \underset{(a)}{(q-1)} = q^3 - q$.

Complete intersections

The Hodge numbers of a smooth hypersurface $X \subset \mathbb{P}^n$ of degree d can be computed recursively from exact sequences

$$0 \to \Omega^p_{\mathbb{P}^n}(-d) \to \Omega^p_{\mathbb{P}^n} \to \Omega^p_{\mathbb{P}^n}|_X \to 0$$

$$0 \to \Omega^{p-1}_{\mathbb{P}^n}(-d) \to \Omega^p_{\mathbb{P}^n}|_X \to \Omega^p_X \to 0$$

and cohomology of $\mathbb{P}^n$. This can be generalized as in [2] to compute the Hodge numbers of a smooth complete intersection $X \subset \mathbb{P}^n$ from the degrees d_i of the hypersurfaces.

Setup: let $X \subset \mathbb{A}^n$ be the variety with ideal $I = (f_1, \ldots, f_k)$. Recursively compute $e(X)$ as follows:

Base cases

if $1 \in I$ then $e(X) = e(\varnothing) = 0$
if $I = (0)$ then $e(X) = e(\mathbb{A}^n) = q^n$

Product varieties

if $F_1 = \{f_1, \ldots, f_m\}$ and $F_2 = \{f_{m+1}, \ldots, f_k\}$ do not share variables, then $X = X_1 \times X_2$, hence $e(X) = e(X_1) \cdot e(X_2)$

Factor equations

if $f_i = gh$ with g, h non-constant, then

$$e(X) = e(X \cap \{g = 0\}) + e(X \cap \{h = 0\}) - e(X \cap \{g = h = 0\})$$

Linear equations

if $f_i = xg + h$ with g, h not containing x, then let Y be given by the f_j, for $j \neq i$, where x substituted for $-h/g$. Then

$$e(X) = e(X \cap \{g = 0\}) + e(Y) - e(Y \cap \{g = 0\})$$

Blowups

if the singular locus $Z \subset X$ is non-empty, blow up X at Z, given by affine patches U_i and exceptional divisor E. Then

$$e(X) = e(Z) + \sum_i e\left(U_i - \bigcup_{j<i} U_j - E \right)$$

Rehomogenizing

if X is non-singular, but the projective closure $\overline{X} \subset \mathbb{P}^n$ is singular at another affine patch Y then

$$e(X) = e(Y) + e\left(\overline{X} - Y\right) - e\left(\overline{X} - X\right)$$

Smooth projective varieties

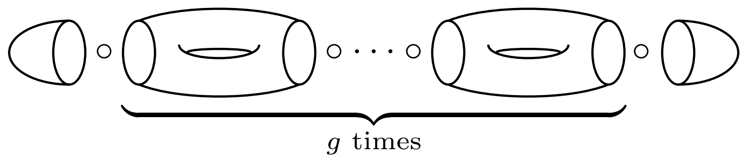
if X defines a smooth projective variety $\overline{X} \subset \mathbb{P}^n$, compute $e(X)$ from the Hodge numbers $h^{p,q}(X) = \dim H^q(X, \Omega^p_X)$

Application to representation varieties

Used in [3] to automize the computation of E-polynomials of G-representation varieties of closed surfaces

$$\mathfrak{X}_G(\Sigma_g, G) = \mathrm{Hom}(\pi_1(\Sigma_g), G)$$

using Topological Quantum Field Theory: the E-polynomials can be obtained from the powers of a (large) matrix of E-polynomials of smaller varieties, corresponding to a decomposition of bordisms



For $G = \mathbb{U}_n$ upper triangular matrices of ranks $2, 3$ and 4:

$$e(\mathfrak{X}_{\mathbb{U}_2}(\Sigma_g)) = q^{2g-1}(q-1)^{2g+1}((q-1)^{2g-1} + 1),$$

$$e(\mathfrak{X}_{\mathbb{U}_3}(\Sigma_g)) = q^{3g-3}(q-1)^{2g}\big(q^2(q-1)^{2g+1} + q^{3g}(q-1)^2 + q^{3g}(q-1)^{4g} + 2q^{3g}(q-1)^{2g+1}\big),$$

$$\begin{aligned} e(\mathfrak{X}_{\mathbb{U}_4}(\Sigma_g)) = {} & q^{8g-2}(q-1)^{4g+2} + q^{8g-2}(q-1)^{6g+1} \\ & + q^{10g-4}(q-1)^{2g+3} + q^{10g-4}(q-1)^{4g+1}\big(2q^2 - 6q + 5\big)^g \\ & + 3q^{10g-4}(q-1)^{4g+2} + q^{10g-4}(q-1)^{6g+1} + q^{12g-6}(q-1)^{8g} \\ & + q^{12g-6}(q-1)^{2g+3} + 3q^{12g-6}(q-1)^{4g+2} + 3q^{12g-6}(q-1)^{6g+1}, \end{aligned}$$

the latter requiring to evaluate ≈ 4000 E-polynomials.

What's next?

- Find more efficient methods for computing the Hodge numbers for non-complete intersections
- Prove the algorithm terminates, e.g. find a numerical invariant that decreases at each step
- Optimize the implementation

References

[1] Deligne, P., Théorie de Hodge III. Inst. Hautes Études Sci. Publ. Math. No. 44 (1974)
[2] SGA7 exposé XI, théorème 2.3
[3] Hablicsek, M., Vogel, J., Virtual classes of representation varieties of upper triangular matrices via topological quantum field theories (2020) arXiv:2008.06679